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A MIXED FINITE ELEMENT METHOD FOR A SIXTH ORDER ELLIPTIC PROBLEM

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Abstract. We consider a saddle point formulation for a sixth order partial differential equation and its finite element approximation, for two sets of boundary conditions. We follow the Ciarlet-Raviart formulation for the biharmonic problem to formulate our saddle point problem and the finite element method. The new formulation allows us to use the H^1 -conforming Lagrange finite element spaces to approximate the solution. We prove a priori error estimates for our approach. Numerical results are presented for linear and quadratic finite element methods.

Key words. Sixth order problem, higher order partial differential equations, biharmonic problem, mixed finite elements, error estimates.

AMS subject classifications. 65N30, 65N15, 35J35 (Primary) 35J40 (Secondary)

1. Introduction. Partial differential equations (PDE) have a long and rich history of application in physical problems. One of their main advantages is in the modelling of ideal or desired structures [36]. In particular, one may wish to fill a curve with a solid material that satisfies certain conditions along the boundary. Depending on the application, there may be several constraints along the curve. In many applications these filled curves (called *components*) are fitted together to form a larger shape. It is natural and in some situations essential that at least some of the derivatives of the surface are continuous across the boundary curves.

In this context, higher-order partial differential equations come to the fore: for a solution of a partial differential equation of order 2k, one may typically allow restrictions on all derivatives up to order (k-1) along the boundary curve. This guarantees their continuity across components.

Continuity of the second derivative across boundaries, achieved by the sixth-order PDE proposed in this article, is critical in several settings. In the construction of automobiles, each panel is designed by a computer based on given specifications. Aesthetics are an important aspect, and in this regard, the composition of reflections from the surface of a car panel must be considered. If one prescribes only the derivatives up to first order along the boundary, then this leaves open the possibility of the second derivative of the panel changing sign across the boundary. In practical terms, this causes boundaries to move from being convex to concave, or vice-versa. Reflections will flip across such boundaries, which from an aesthetic perspective is unacceptable.

The strength and maximal load bearing of tensile structures also depends critically on the continuity of higher derivatives across component boundaries. Force is optimally spread uniformly across components, however, where derivatives of the surface are large, force and load are accumulated. This can be by design. It is dangerous however when force accumulates across a boundary due not to design but to a discontinuity in one of the higher derivatives across that boundary. This concern can be alleviated when a number of derivatives dependent upon the total expected load of the structure can be guaranteed to be continuous. Two derivatives are guaranteed by our scheme and this is typically enough for most minor structures, such as small buildings, residential homes, and vehicles.

Sixth-order PDE have arisen in a variety of other contexts, from propeller blade design [13] to ulcer modelling [33]. Generic applications of sixth-order PDE to manufacturing are mentioned in

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[3, 5]. Applications of sixth order problems in surface modelling and fluid flows are considered in [25, 31].

To see that sixth-order PDE are natural for such applications, it is instructive to view such an equation variationally. Minimising the classical Dirichlet energy, we calculate the first variation of the functional

$$\int_{\Omega} |\nabla u|^2 dx \,,$$

and find the Laplace equation

 $\Delta u=0$

or, in the case of the gradient flow, the heat equation

$$(\partial_t - \Delta)u = 0$$
.

Minimising the elastic energy, the integrand of the functional to be minimised depends on an additional order of derivative of u, and so the Euler-Lagrange equation and resulting gradient flow is of fourthorder. If we are additionally interested in minimising the rate of change of curvature across the surface, the 'rate of change of acceleration' or *jerk*, then the functional will depend on three orders of derivatives of u. The resulting Euler-Lagrange equation

$$\Delta^3 u = 0$$

and gradient flow

$$(\partial_t - \Delta^3)u = 0$$

depend on six orders of derivatives of u. This perspective is taken in Section 2, where the variational formulation is made rigorous. Recent resarch interest in such equations includes [18, 19, 26, 28].

In geophysics, sixth-order PDE are used to overcome difficulties involving complex geological faults [35]. Indeed, sixth-order PDE arise in a variety of geophysical contexts due to their appearance as models in electromagneto-thermoelasticity [30] and relation to equatorial electrojets [34]. We remark that model PDE from geophysics are in general quite interesting to study from a PDE perspective, with issues such as non-uniqueness and general ill-posedness fundamental characteristics; we refer to [24] for a selection of such issues.

The major contribution in our paper is a mixed finite element scheme for a sixth-order partial differential equation. This allows one to accurately model components arising from prescribed (up to and including) second order derivatives along boundary curves. Another approach to approximate the solution of the sixth-order elliptic problem based on the interior penalty is considered by Gudi and Neilan [17]. In Section 2 we introduce our setting, which considers two different sets of boundary conditions: simply supported, and clamped. We use constrained minimisation to cast our problems in a mixed formulation as in the case of the biharmonic equation [9, 12, 21] (other approaches to mixed formulations for the biharmonic equation can be found in [11, 10, 14, 15, 2, 27, 22]). The resulting saddle point problem allows us to apply low order H^1 -conforming finite element methods to approximate the solution of the sixth order problem. This approximation is described, for both sets of boundary conditions, in Section 3. A-priori error estimates are proved in Section 4. The optimality of the predicted rates of convergences is illustrated, for each boundary condition, in Section 5 through various numerical results.

2. A mixed formulation of a sixth order elliptic equation. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with polygonal or polyhedral boundary $\partial \Omega$ and outward pointing normal n on $\partial \Omega$. We consider the sixth order problem

$$-\Delta^3 u = f \quad \text{in} \quad \Omega \tag{2.1} \quad \text{biharm}$$

with $f \in H^{-1}(\Omega)$ and two sets of boundary conditions (BCs). The first set is the set of simply supported boundary conditions

$$u = \Delta u = \Delta^2 u = 0$$
 on $\partial \Omega$, (2.2) sate

and the second set is the set of *clamped* boundary conditions

$$u = \frac{\partial u}{\partial n} = \Delta u = 0 \quad \text{on} \quad \partial \Omega.$$
 (2.3) clabc

We aim at obtaining a formulation only based on the H^1 -Sobolev space. We begin by defining the Lagrange multiplier space:

• Simply supported boundary conditions. We set

$$M_{bc} = H_0^1(\Omega)$$

and equip M_{bc} with the norm

 $||v||_{M_{bc}} = ||v||_{1,\Omega}.$

• Clamped boundary conditions. We set

$$M_{bc} = \{ q \in H^{-1}(\Omega) : \Delta q \in H^{-1}(\Omega) \},\$$

where Δq is interpreted in the distributional sense, and the space M_{bc} is equipped with the graph norm

$$\|q\|_{M_{bc}} = \sqrt{\|q\|_{-1,\Omega}^2 + \|\Delta q\|_{-1,\Omega}^2}.$$

We use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing between the two spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, so that $\langle u, q \rangle$ and $\langle u, \Delta q \rangle$ are well defined for $u \in H_0^1(\Omega)$ and $q \in M_{bc}$. We note that this space M_{bc} is less regular than $H^1(\Omega)$, compare, e.g., [4, 37].

Let $k \in \mathbb{N} \cup \{0\}$. We use the standard notations to represent Sobolev spaces [1, 8]. We use $(\cdot, \cdot)_{k,\Omega}$ and $\|\cdot\|_{k,\Omega}$ to denote the inner product and norm in $H^k(\Omega)$, respectively. When k = 0, we get the inner product $(\cdot, \cdot)_{0,\Omega}$ and the norm $\|\cdot\|_{0,\Omega}$ in $L^2(\Omega)$. The norm of $W^{k,p}(\Omega)$ is denoted by $\|\cdot\|_{k,p,\Omega}$.

To obtain the H^1 -based formulation of our boundary value problems, we introduce an additional unknown $\phi = \Delta u$ and write a weak form of this equation by formally multiplying by a function $q \in M_{bc}$ and integrating over Ω , as in [4, 37]. The variational equation is now written as

$$\langle \phi, q \rangle - \langle u, \Delta q \rangle = 0, \quad q \in M_{bc}$$

Keeping in mind that u will be taken in $H_0^1(\Omega)$, and considering the BC-dependent M_{bc} , we see that this variational definition of " $\phi = \Delta u$ " also formally imposes the condition $\frac{\partial u}{\partial n} = 0$ on $\partial \Omega$, in the case of clamped BCs. For simply supported BCs, this does not impose any additional boundary conditions.

To write the mixed formulation in a standard setting, we introduce the function space V = $H_0^1(\Omega) \times H_0^1(\Omega)$ with the inner product $(\cdot, \cdot)_V$ defined as

$$((u,\phi),(v,\psi))_V = (\nabla u,\nabla v)_{0,\Omega} + (\nabla \phi,\nabla \psi)_{0,\Omega}$$

and with the norm $\|\cdot\|_V$ induced by this inner product. We now consider the constraint minimisation problem of finding $(u, \phi) \in \mathcal{V}$ such that

$$\mathcal{J}(u,\phi) = \inf_{(v,\psi)\in\mathcal{V}} \mathcal{J}(v,\psi), \tag{2.4}$$
 mbiharm

where

$$\mathcal{J}(v,\psi) = \frac{1}{2} \int_{\Omega} |\nabla\psi|^2 \, dx - \langle f, v \rangle, \quad \text{and}$$

$$\mathcal{V} = \{(v,\psi) \in V : \langle \psi, q \rangle - \langle u, \Delta q \rangle = 0, \ q \in M_{bc} \}.$$

$$(2.5) \quad \text{def:CV}$$

Looking for (u, ψ) in V enables us to account for the conditions $u = \Delta u = 0$ on $\partial \Omega$, valid for both simply supported and clamped BCs.

The problem (2.4) can be recast as a saddle point formulation: find $((u, \phi), \lambda) \in V \times M_{bc}$ so that

$$\begin{array}{rcl} a((u,\phi),(v,\psi)) + & b((v,\psi),\lambda) &= & \ell(v), & (v,\psi) \in V, \\ b((u,\phi),\mu) &= & 0, & \mu \in M_{bc}, \end{array}$$
 (2.6) wbiharm

where

$$a((u,\phi),(v,\psi)) = \int_{\Omega} \nabla \phi \cdot \nabla \psi \, dx, \quad b((v,\psi),\mu) = \langle \psi,\mu \rangle - \langle v,\Delta\mu \rangle,$$

$$\ell(v) = \langle f,v \rangle.$$
(2.7) defbiharm

Using v = 0 and $\psi \in C_c^{\infty}(\Omega)$ in the first equation in (2.6) shows that $\Delta \phi = \lambda$. In the case of simply supported boundary conditions, since $\lambda \in (M_{bc}) = H_0^1(\Omega)$ and $\phi = \Delta u$, this enables us to formally recover the last missing boundary condition $\Delta^2 u = 0$ on $\partial \Omega$.

The following theorem, whose proof can be found in the appendix, states the well-posedness of our continuous saddle point problem.

THEOREM 2.1. There exists a unique $((u, \phi), \lambda) \in V \times M_{bc}$ satisfying (2.6).

3. Finite element discretisations. We consider a quasi-uniform and shape-regular triangulation \mathcal{T}_h of the polygonal domain Ω , where \mathcal{T}_h consists of triangles, tetrahedra, parallelograms or hexahedra. Let $S_h^k \subset H^1(\Omega)$ be a standard Lagrange finite element space of degree $k \geq 1$ based on the triangulation \mathcal{T}_h with the following approximation property: For $u \in H^{k+1}(\Omega)$

$$\inf_{v_h \in S_h^k} (\|u - v_h\|_{0,\Omega} + h\|u - v_h\|_{1,\Omega}) \le Ch^{k+1} \|u\|_{k+1,\Omega}.$$
(3.1) approx0

The definition of discrete Lagrange multiplier spaces $(M_{bc})_h^k$ requires some work. A standard requirement for the construction is the following list of properties:

[P1] $(M_{bc})_h^k \subset H^1(\Omega)$. [P2] There is a constant C independent of the triangulation such that

$$\|\theta_h\|_{0,\Omega} \le C \sup_{\phi_h \in S_{h,0}^k} \frac{\int_{\Omega} \theta_h \phi_h \, dx}{\|\phi_h\|_{0,\Omega}}, \quad \theta_h \in (M_{bc})_h^k.$$

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[P3] There is a constant C independent of the triangulation such that, if (u, ϕ, λ) is a solution to (2.6), $\lambda \in H^k(\Omega)$ and $\mu_h \in (M_{bc})_h^k$ is the H^1 -orthogonal projection of λ on $(M_{bc})_h^k$, then

$$\|\lambda - \mu_h\|_{0,\Omega} \le Ch^k \|\lambda\|_{k,\Omega}.$$
(3.2) approx1

We now define:

• Simply supported boundary conditions. In this case we may simply take

$$S_{h,0}^k = S_h^k \cap H_0^1(\Omega), \quad V_h^k = S_{h,0}^k \times S_{h,0}^k, \quad (M_{bc})_h^k = S_{h,0}^k.$$

The norm on $(M_{bc})_h^k$ is defined by

$$\|\mu_h\|_h = \sqrt{\|\mu_h\|_{-1,h}^2 + \|\Delta\mu_h\|_{-1,h}^2} \quad \text{with} \quad \|\mu_h\|_{-1,h} = \sup_{v_h \in S_{h,0}^k} \frac{\langle \mu_h, v_h \rangle}{\|\nabla v_h\|_{0,\Omega}}.$$

The reader may wish to compare this with [37], where a similar norm is used albeit with $\mu_h \in L^2(\Omega)$. Properties [P1] and [P2] are trivial. Property [P3] is establised by invoking the fact that $\lambda = 0$ on $\partial \Omega$, and by using the approximation results in [6, 7].

Clamped boundary conditions. The first two spaces are

$$S_{h,0}^k = S_h^k \cap H_0^1(\Omega) \,, \quad V_h^k = S_{h,0}^{k+1} \times S_{h,0}^k \,,$$

however the space $(M_{bc})_h^k$ is not so easily defined. If we take $(M_{bc})_h^k = S_{h,0}^k$, the Lagrange multiplier space does not have the required approximation property, due to the constraint on the boundary condition. On the other hand, if we take $(M_{bc})_h^k = S_h^k$, the stability assumption [P2] will be lost.

In order to overcome this we draw inspiration from the idea used in the mortar finite element method [23, 20]: We construct the Lagrange multiplier space $(M_{bc})_h^k$ satisfying dim $(M_{bc})_h^k =$ dim $S_{h,0}^k$ and the approximation property (3.2). To construct the basis functions of $(M_{bc})_h^k$ for the clamped boundary condition we start with S_h^k and remove all basis functions of S_h^k associated with the boundary of the domain Ω . We construct the basis functions of $(M_{bc})_{h}^{k}$ according to the following steps:

- 1. For a basis function φ_n of S_h^k associated with the point x_n on the boundary we find a closest internal triangle/tetrahedron/parallelotope $T \in \mathcal{T}_h$ (that is, T does not touch $\partial \Omega$).
- 2. The basis functions $\{\varphi_{T,i}\}_{i=1}^m$ associated with internal points of T can be considered as polynomials defined on the whole domain Ω . Hence we can compute $\{\alpha_{T,i}\}_{i=1}^m$ as $\alpha_{T,i} = \varphi_{T,i}(x_n)$ for $i = 1, \cdots, m$. This means when computing $\{\alpha_{T,i}\}_{i=1}^m$ we regard $\{\varphi_{T,i}\}_{i=1}^{m}$ as polynomials with support on $\overline{\Omega}$. For the linear finite element, the coefficients $\{\alpha_{T,i}\}_{i=1}^{m}$ are the barycentric coordinates of x_n with respect to T. 3. Then we modify all the basis functions $\{\varphi_{T,i}\}_{i=1}^{m}$ associated with T as $\tilde{\varphi}_{T,i} = \varphi_{T,i} + \varphi_{T,i}$
- $\alpha_T \,_i \varphi_n$.

In other words, basis functions associated with boundary points are "re-distributed" on basis functions associated with nearby internal points, which ensures that, even after removing these boundary basis functions, the space $(M_{bc})_h^k$ has the same approximation property as S_h^k . The norm on $(M_{bc})_h^k$ is defined by

$$\|\mu_h\|_h = \sqrt{\|\mu_h\|_{-1,h}^2 + \|\Delta\mu_h\|_{-1,h*}^2} \quad \text{with} \quad \|\mu_h\|_{-1,h*} = \sup_{v_h \in S_{h,0}^{k+1}} \frac{\langle \mu_h, v_h \rangle}{\|\nabla v_h\|_{0,\Omega}}.$$

Then [P2] and the optimal approximation property (3.2) follow (see [20, 23]).

In the following, we use a generic constant C, which takes different values in different occurrences but is always independent of the mesh-size. Now, the finite element problem is to find $((u_h, \phi_h), \lambda_h) \in V_h^k \times (M_{bc})_h^k$ so that

$$\begin{array}{ll} a_h((u_h,\phi_h),(v_h,\psi_h)) + & b((v_h,\psi_h),\lambda_h) &= & \ell(v_h), & (v_h,\psi_h) \in V_h^k, \\ b((u_h,\phi_h),\mu_h) &= & 0, & \mu_h \in (M_{bc})_h^k. \end{array}$$
(3.3) weakh

For simply supported BCs, we can take $a_h = a$. For the case of clamped boundary conditions, $S_{h,0}^k$ is not contained in $(M_{bc})_h^k$, and so $a_h(\cdot, \cdot)$ is a stabilised form of the bilinear form a. This allows us to establish coercivity (see the proof of Theorem 3.2 below). We set $a_h(\cdot, \cdot)$ to be

$$a_h((u_h,\phi_h),(v_h,\psi_h)) = a((u_h,\phi_h),(v_h,\psi_h)) + \int_{\Omega} (\phi_h - \Delta_h u_h)(\psi_h - \Delta_h v_h) \, dx, \qquad (3.4) \quad \text{EQstabilised}$$

where, for $w \in H_0^1(\Omega) + S_{h,0}^{k+1}$, $\Delta_h w \in S_{h,0}^{k+1}$ is given by

$$\int_{\Omega} \Delta_h w \, v_h \, dx = -\int_{\Omega} \nabla w \cdot \nabla v_h \, dx, \quad v_h \in S_{h,0}^{k+1}. \tag{3.5}$$

REMARK 3.1. For simply supported BCs, that is $a_h = a$, the saddle point problem (3.3) can be, as with the continuous problem, recast in the form of a constraint minimisation problem: find $(u_h, \phi_h) \in \mathcal{V}_h^k$ such that

$$\mathcal{J}(u_h,\phi_h) = \inf_{(v_h,\psi_h)\in\mathcal{V}_h^k} \mathcal{J}(v_h,\psi_h), \tag{3.6}$$
 cmbiharmh

where \mathcal{V}_{h}^{k} is a kernel space defined as

$$\mathcal{V}_{h}^{k} = \{ (v_{h}, \psi_{h}) \in V_{h}^{k} : b((v_{h}, \psi_{h}), \mu_{h}) = 0, \ \mu_{h} \in (M_{bc})_{h}^{k} \}.$$
(3.7) def:CV

We now show the existence of a unique solution to (3.3).

THEOREM 3.2. There exists a unique $(u_h, \phi_h) \in V_h^k$ solution to (3.3).

Proof. Existence of a unique solution to (3.3) relies on the same three properties as in the continuous case, namely:

- 1. The bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and the linear form $\ell(\cdot)$ are uniformly continuous on $V_h^k \times V_h^k$, $V_h^k \times (M_{bc})_h^k$ and V_h^k , respectively. The bilinear form $a_h(\cdot, \cdot)$ is continuous (albeit not uniformly) on $V_h^k \times V_h^k$. Here, V_h^k is endowed with the norm of V, and $(M_{bc})_h^k$ with its norm $\|\cdot\|_h$.
- 2. The bilinear form $a_h(\cdot, \cdot)$ is uniformly coercive on the kernel space \mathcal{V}_h^k defined by (3.7).
- 3. The bilinear form $b(\cdot, \cdot)$ satisfies the following inf-sup condition

$$\inf_{\mu_h \in (M_{bc})_h^k} \sup_{(v_h, \psi_h) \in V_h^k} \frac{b((v_h, \psi_h), \mu_h)}{\|(v_h, \psi_h)\|_V \|\mu_h\|_h} \ge \tilde{\beta},$$

where $\tilde{\beta}$ is a constant independent of the mesh-size.

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Since $V_h^k \subset V$, the uniform continuities of $a(\cdot, \cdot)$ and $\ell(\cdot)$ are trivial. The continuity of $a_h(\cdot, \cdot)$ on the finite dimensional space V_h^k is obvious. However, since we cannot claim that $\|\Delta_h v\|_{0,\Omega} \leq C \|v\|_{1,\Omega}$ with C independent on h, this continuity of $a_h(\cdot, \cdot)$ is not uniform; this is not required to obtain the existence and uniqueness of a solution to the scheme, but it will force us to define a stronger, mesh-dependent norm for the convergence analysis (see Section 4.2). The uniform continuity of the bilinear form $b(\cdot, \cdot)$ is proved as follows. Note that since $\psi_h \in S_{h,0}^k$ we have from the definition of $\|\cdot\|_{-1,h}$ - norm

$$\|\mu_h\|_{-1,h} \|\nabla \psi_h\|_{0,\Omega} = \sup_{v_h \in S_{h,0}^k} \frac{\int_{\Omega} v_h \, \mu_h \, dx}{\|\nabla v_h\|_{0,\Omega}} \|\nabla \psi_h\|_{0,\Omega} \ge \int_{\Omega} \psi_h \, \mu_h \, dx.$$

For the simply supported case with $v_h \in S_{h,0}^k$ we have

$$\|\Delta\mu_h\|_{-1,h} \|\nabla v_h\|_{0,\Omega} = \sup_{w_h \in S_{h,0}^k} \frac{\int_{\Omega} \nabla w_h \cdot \nabla \mu_h \, dx}{\|\nabla w_h\|_{0,\Omega}} \|\nabla v_h\|_{0,\Omega} \ge \int_{\Omega} \nabla v_h \cdot \nabla \mu_h \, dx \,,$$

whereas for the clamped case with $v_h \in S_{h,0}^{k+1}$ we have

$$\|\Delta\mu_h\|_{-1,h*} \|\nabla v_h\|_{0,\Omega} = \sup_{w_h \in S_{h,0}^{k+1}} \frac{\int_{\Omega} \nabla w_h \cdot \nabla \mu_h \, dx}{\|\nabla w_h\|_{0,\Omega}} \|\nabla v_h\|_{0,\Omega} \ge \int_{\Omega} \nabla v_h \cdot \nabla \mu_h \, dx \, .$$

The continuity of $b(\cdot, \cdot)$ follows by writing

$$|b((v_h, \psi_h), \mu_h)| = \left| \langle \psi_h, \mu_h \rangle - \langle v_h, \Delta \mu_h \rangle \right| \le \left| \int_{\Omega} \psi_h \, \mu_h \, dx + \int_{\Omega} \nabla v_h \cdot \nabla \mu_h \, dx \right|$$
$$\le \|(v_h, \psi_h)\|_V \|\mu_h\|_h.$$

This establishes the first condition. For the second and third condition we now must consider the boundary conditions separately.

Simply supported boundary conditions. For $(u_h, \phi_h) \in V_h^k$ satisfying

$$b((u_h, \phi_h), \mu_h) = 0, \quad \mu_h \in (M_{bc})_h^k,$$

since $(M_{bc})_{h}^{k} = S_{h,0}^{k}$, we can take $\mu_{h} = u_{h}$ to obtain

$$\int_{\Omega} \nabla u_h \cdot \nabla u_h \, dx = -\int_{\Omega} \phi_h u_h \, dx.$$

Hence using the Cauchy-Schwarz and Poincaré inequalities we obtain

$$\|\nabla u_h\|_{0,\Omega}^2 \le C_1 \|\phi_h\|_{0,\Omega} \|\nabla u_h\|_{0,\Omega}.$$

The coercivity then follows exactly as in the continuous case:

$$\|\nabla u_h\|_{0,\Omega}^2 + \|\nabla \phi_h\|_{0,\Omega}^2 \le Ca((u_h, \phi_h), (u_h, \phi_h)), \quad (u_h, \phi_h) \in \mathcal{V}_h^k.$$

For the inf-sup condition we set $\psi_h = 0$ as in the continuous setting to obtain

$$\sup_{(v_h,\psi_h)\in V_h^k} \frac{b((v_h,\psi_h),\mu_h)}{\|(v_h,\psi_h)\|_V} \ge \sup_{v_h\in S_{h,0}^k} \frac{\langle v_h,\Delta\mu_h\rangle}{\|\nabla v_h\|_{0,\Omega}} \ge \|\Delta\mu_h\|_{-1,h},$$

and setting $v_h = 0$ to find

$$\sup_{(v_h,\psi_h)\in V_h^k} \frac{b((v_h,\psi_h),\mu_h)}{\|(v_h,\psi_h)\|_V} \ge \sup_{\psi_h\in S_{h,0}^k} \frac{\langle \psi_h,\mu_h \rangle}{\|\nabla \psi_h\|_{0,\Omega}} \ge \|\mu_h\|_{-1,h}.$$

Thus

$$\sup_{(v_h,\psi_h)\in V_h^k} \frac{b((v_h,\psi_h),\mu_h)}{\|(v_h,\psi_h)\|_V} \ge \tilde{\beta} \|\mu_h\|_h.$$

Clamped boundary conditions. Recalling the stabilisation term in $a_h(\cdot, \cdot)$, we use the Poincaré inequality for $u_h \in S_{h,0}^{k+1}$ and the definition (3.5) of Δ_h to find

$$\begin{aligned} \|\nabla u_h\|_{0,\Omega} &= \sup_{v_h \in S_{h,0}^{k+1}} \frac{\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx}{\|\nabla v_h\|_{0,\Omega}} \le C \sup_{v_h \in S_{h,0}^{k+1}} \frac{\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx}{\|v_h\|_{0,\Omega}} \\ &= C \sup_{v_h \in S_{h,0}^{k+1}} \frac{-\int_{\Omega} \Delta_h u_h \, v_h \, dx}{\|v_h\|_{0,\Omega}} \le C \|\Delta_h u_h\|_{0,\Omega}. \end{aligned}$$

Hence using Poincaré inequality again, there exists a positive constant C such that, for all $\phi_h \in S_{h,0}^k$,

$$\|\nabla u_h\|_{0,\Omega}^2 \le C\left(\|\phi_h - \Delta_h u_h\|_{0,\Omega}^2 + \|\phi_h\|_{0,\Omega}^2\right) \le C\left(\|\phi_h - \Delta_h u_h\|_{0,\Omega}^2 + \|\nabla\phi_h\|_{0,\Omega}^2\right).$$

Thus we have the coercivity of the modified bilinear form $a_h(\cdot, \cdot)$ on $S_{h,0}^{k+1} \times S_{h,0}^k$ and hence on the discrete kernel space $\mathcal{V}_h^k \subset S_{h,0}^{k+1} \times S_{h,0}^k$ with respect to the standard norm of V. The inf-sup condition now follows as in the case of simply supported boundary conditions, with $S_{h,0}^{k+1}$ instead of $S_{h,0}^k$ for v_h , which accounts for $\|\cdot\|_{-1,h*}$ used in the definition of the norm on $(M_{bc})_h^k$. This finishes the proof of the theorem. \Box

4. A priori error estimates. In this section we investigate a priori error estimates for our problems.

4.1. A priori error estimate for simply supported boundary conditions. Our goal is to establish the following theorem.

THEOREM 4.1. Let (u, ϕ, λ) be the solution of the saddle point problem (2.6), and (u_h, ϕ_h, λ_h) the solution of (3.3), both with simply supported boundary conditions. We assume that $u, \phi \in H^{k+1}(\Omega)$ and $\lambda \in H^k(\Omega)$. Then

$$\|(u - u_h, \phi - \phi_h)\|_V \le Ch^k \left(\|u\|_{k+1,\Omega} + \|\phi\|_{k+1,\Omega} + |\lambda|_{k,\Omega}\right).$$
(4.1) |eq:strang1

To prove this theorem we apply Strang's second lemma [7]:

$$\|(u - u_h, \phi - \phi_h)\|_{V} \leq C \left(\inf_{(v_h, \psi_h) \in \mathcal{V}_h^k} \|(u - v_h, \phi - \psi_h)\|_{V} + \sup_{(v_h, \psi_h) \in \mathcal{V}_h^k} \frac{|a((u - u_h, \phi - \phi_h), (v_h, \psi_h))|}{\|(v_h, \psi_h)\|_{V}} \right), \quad (4.2) \quad \text{eq:strang}$$

8

ec:error.est

th0

where (u, ϕ) is the solution of (2.4), and (u_h, ϕ_h) the solution of (3.3) (recall that, here, $a_h = a$). The first term in the right side of (4.2) is the best approximation error and the second one stands for the consistency error. First we turn our attention to this latter term.

LEMMA 4.2. Let (u, ϕ, λ) be the solution of the saddle point problem (2.6) with simply supported boundary conditions. Then, if $\lambda \in H^k(\Omega)$, we have

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u-u_h,\phi-\phi_h),(v_h,\psi_h))|}{\|(v_h,\psi_h)\|_V} \le Ch^k |\lambda|_{k,\Omega}.$$

Proof. From the first equation of (2.6) we get $a((u - u_h, \phi - \phi_h), (v_h, \psi_h)) + b((v_h, \psi_h), \lambda) = 0$ for all $(v_h, \psi_h) \in \mathcal{V}_h^k$. And hence

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u-u_h,\phi-\phi_h),(v_h,\psi_h))|}{\|(v_h,\psi_h)\|_V} = \sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|b((v_h,\psi_h),\lambda)|}{\|(v_h,\psi_h)\|_V}.$$

Denoting the projection of λ onto $(M_{bc})_h^k = S_{h,0}^k$ with respect to the H^1 -inner product by $\tilde{\lambda}_h$, we have

$$\int_{\Omega} \nabla v_h \cdot \nabla (\lambda - \tilde{\lambda}_h) \, dx = -\int_{\Omega} v_h (\lambda - \tilde{\lambda}_h) \, dx. \tag{4.3}$$
 estorth

As $(v_h, \psi_h) \in \mathcal{V}_h^k$, using (4.3),

$$b((v_h,\psi_h),\lambda) = b((v_h,\psi_h),\lambda - \tilde{\lambda}_h) = -\int_{\Omega} v_h(\lambda - \tilde{\lambda}_h) \, dx + \int_{\Omega} \psi_h(\lambda - \tilde{\lambda}_h) \, dx,$$

and hence [P3] yields

$$|b((v_h, \psi_h), \lambda)| \le Ch^k |\lambda|_{k,\Omega} \, \|(v_h, \psi_h)\|_V.$$

Thus

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u-u_h,\phi-\phi_h),(v_h,\psi_h))|}{\|(v_h,\psi_h)\|_V} \le Ch^k |\lambda|_{k,\Omega}$$

	_	-		
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	-	-		

lemma3

We now prove the following lemma, which is similar to Proposition 3 in [12]. See also [21].

LEMMA 4.3. Let $(w_h, \xi_h) \in \mathcal{V}_h^k$, $(w, \xi) \in \mathcal{V}$, and $R_h^k : H_0^1(\Omega) \to S_{h,0}^k$ be the Ritz projector (also called "elliptic projector") defined as

$$\int_{\Omega} \nabla (R_h^k w - w) \cdot \nabla v_h \, dx = 0, \ v_h \in S_{h,0}^k.$$

Then

$$|w - w_h|_{1,\Omega} \le C \|\xi - \xi_h\|_{0,\Omega} + |R_h^k w - w|_{1,\Omega}$$

Proof. Here we have

$$\int_{\Omega} \nabla w \cdot \nabla q + \xi \, q \, dx = 0, \ q \in H_0^1(\Omega), \text{ and } \int_{\Omega} \nabla w_h \cdot \nabla q_h + \xi_h \, q_h \, dx = 0, \ q_h \in S_{h,0}^k,$$

lemma2

since $(w_h, \xi_h) \in \mathcal{V}_h^k$ and $(w, \xi) \in \mathcal{V}$. Thus, since $S_{h,0}^k \subset H_0^1(\Omega)$,

$$\int_{\Omega} \nabla(w - w_h) \cdot \nabla q_h + (\xi - \xi_h) q_h \, dx = 0, \ q_h \in S_{h,0}^k.$$
(4.4) eq2

In terms of the Ritz-projector R_h^k , (4.4) is written as

$$\int_{\Omega} \nabla (R_h^k w - w_h) \cdot \nabla q_h + (\xi - \xi_h) q_h \, dx = 0, \ q_h \in S_{h,0}^k.$$
(4.5) eq3

Taking $q_h = R_h^k w - w_h$ in equation (4.5) and using the Cauchy-Schwarz and Poincaré inequalities, we obtain

$$|R_h^k w - w_h|_{1,\Omega}^2 \le \|\xi - \xi_h\|_{0,\Omega} \|R_h^k w - w_h\|_{0,\Omega} \le C \|\xi - \xi_h\|_{0,\Omega} |R_h^k w - w_h|_{1,\Omega},$$

which yields $|R_h^k w - w_h|_{1,\Omega} \leq C \|\xi - \xi_h\|_{0,\Omega}$. The final result follows from the triangle inequality

$$|w - w_h|_{1,\Omega} \le |R_h^k w - w_h|_{1,\Omega} + |w - R_h^k w|_{1,\Omega} \le C ||\xi - \xi_h||_{0,\Omega} + |w - R_h^k w|_{1,\Omega}.$$

The following lemma estimates the best approximation error in (4.2), and concludes the proof of Theorem 4.1. LEMMA 4.4. For any $(u, \phi) \in \mathcal{V} \cap (H^{k+1}(\Omega) \times H^{k+1}(\Omega))$, there exists $(w_h, \psi_h) \in \mathcal{V}_h^k$ such that

lemma12

$$\|(u - w_h, \phi - \xi_h)\|_V \le Ch^k \left(\|u\|_{k+1,\Omega} + \|\phi\|_{k+1,\Omega}\right)$$
(4.6)

Proof. Let $\Pi_h^k : L^2(\Omega) \to S_{h,0}^k$ be the orthogonal projection onto $S_{h,0}^k$. Let $(w_h, \xi_h) \in V_h^k$ be defined as

$$\int_{\Omega} (\phi - \xi_h) q_h dx = 0, \ q_h \in S_{h,0}^k, \text{ and } \int_{\Omega} \nabla w_h \cdot \nabla q_h + \xi_h q_h dx = 0, \ q_h \in S_{h,0}^k$$

Hence $(w_h, \xi_h) \in \mathcal{V}_h^k$ with $\xi_h = \prod_h^k \phi$. Moreover, since \prod_h^k is the L^2 -projection onto $S_{h,0}^k$ we have [6]

$$|\phi - \xi_h|_{1,\Omega} \le Ch^k |\phi|_{k+1,\Omega}.$$

We note that the Ritz projector R_h^k as defined in Lemma 4.3 has the approximation property [32]

$$|u - R_h^k u|_{1,\Omega} \le Ch^k |u|_{k+1,\Omega}.$$

Hence using the result of Lemma 4.3 we obtain

$$|u - w_h|_{1,\Omega} \le \|\phi - \xi_h\|_{0,\Omega} + |u - R_h^k u|_{1,\Omega} \le Ch^k (|u|_{k+1,\Omega} + |\phi|_{k+1,\Omega}).$$

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ence.clamped

4.2. A priori error estimates for clamped boundary conditions. The error estimates for clamped boundary conditions are established in the following mesh-dependent semi-norm: for $(u,\phi) \in V + V_h^k,$

$$|(u,\phi)|_{k,h} = \sqrt{\|\nabla\phi\|_{0,\Omega}^2 + \|\phi - \Delta_h u\|_{0,\Omega}^2}.$$
(4.7) norm

The reason for introducing this semi-norm is that, as already noticed in the proof of Theorem 3.2, the stabilisation term in $a_h(\cdot, \cdot)$ is not uniformly continuous on V_h^k for the norm of V. On the contrary, $a_h(\cdot, \cdot)$ is uniformly continuous for $|\cdot|_{k,h}$, which enables the usage of the second Strang Lemma.

Our goal here is to establish the following a priori estimate.

THEOREM 4.5. Let (u, ϕ, λ) be the solution of the saddle point problem (2.6), and (u_h, ϕ_h, λ_h) the solution of (3.3), both with clamped boundary conditions. We assume that $u \in W^{k+1,p}(\Omega)$ for some $p \geq 2, \phi \in H^{k+1}(\Omega)$ and that $\lambda \in H^k(\Omega)$. We have

$$|(u - u_h, \phi - \phi_h)|_{k,h} \le C \left(h^k ||u||_{k+1,\Omega} + h^{k - \frac{1}{2} - \frac{1}{p}} ||u||_{k+1,p,\Omega} + h^k ||\phi||_{k+1,\Omega} + h^k |\lambda|_{k,\Omega} \right).$$
(4.8) eq:stra

REMARK 4.6. Due to the uniform coercivity property of $a_h(\cdot, \cdot)$ on \mathcal{V}_h^k (see Theorem 3.2), $|\cdot|_{k,h}$ is a norm that is uniformly stronger than the V norm, that is, there is C > 0 independent of h such that, if $(u_h, \phi_h) \in \mathcal{V}_h^k$ then

$$|C|(u_h, \phi_h)|_{k,h} \ge \|\nabla \phi_h\|_{0,\Omega}^2 + \|\nabla u_h\|_{0,\Omega}^2.$$

This property is all that is required to apply the second Strang lemma below. The semi-norm is not a norm on V, but the following property can be established: the kernel of $|\cdot|_{k,h}$ consists of pairs (u,0)such that

$$\int_{\Omega} \nabla u \cdot \nabla u_h \, dx = 0, \quad u_h \in S_{h,0}^{k+1}.$$

Hence, even though the estimate (4.8) might not 'capture' a part of the solution (u, ϕ) , that part actually converges to zero in L^2 and H^1 -norms.

We follow a strategy analogous to that used for simply supported BCs. Even though the second Strang lemma is often used for bilinear forms $a_h(\cdot, \cdot)$ that are coercive on the entire continuous and discrete spaces, the proof of [6, Lemma 1.2, Chap. III, § 1] and the uniform coercivity (by construction) of $a_h(\cdot, \cdot)$ with respect to $|\cdot|_{k,h}$ show that the following estimate holds:

$$|(u - u_h, \phi - \phi_h)|_{k,h} \leq C \left(\inf_{(v_h, \psi_h) \in \mathcal{V}_h^k} |(u - v_h, \phi - \psi_h)|_{k,h} + \sup_{(v_h, \psi_h) \in \mathcal{V}_h^k} \frac{|a_h((u, \phi), (v_h, \psi_h)) - \ell(v_h)|}{|(v_h, \psi_h)|_{k,h}} \right).$$
(4.9) eq:strang

Theorem 4.5 is proved if we bound the right-hand side of the above inequality by the right-hand side of (4.8).

First we prove the following lemma to estimate the consistency error term

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a_h((u,\phi),(v_h,\psi_h)) - \ell(v_h)|}{|(v_h,\psi_h)|_{k,h}}.$$

th1

LEMMA 4.7. Let (u, ϕ, λ) be the solution of the saddle point problem (2.6). Then, if $\lambda \in H^k(\Omega)$, $\phi \in H^k(\Omega)$ and $u \in H^2(\Omega)$ we have

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a_h((u,\phi),(v_h,\psi_h)) - \ell(v_h)|}{|(v_h,\psi_h)|_{k,h}} \le Ch^k \left(|\lambda|_{k,\Omega} + |\phi|_{k,\Omega}\right).$$

Proof. Here

$$a_h((u,\phi),(v_h,\psi_h)) - \ell(v_h) = a((u,\phi),(v_h,\psi_h)) + \int_{\Omega} (\phi - \Delta_h u)(\psi_h - \Delta_h v_h) \, dx - \ell(v_h).$$

The first equation of (2.6) yields

$$a((u,\phi),(v_h,\psi_h)) + b((v_h,\psi_h),\lambda) = \ell(v_h), \quad (v_h,\psi_h) \in V_h^k,$$

and hence

$$a_h((u,\phi),(v_h,\psi_h)) - \ell(v_h) = \int_{\Omega} (\phi - \Delta_h u)(\psi_h - \Delta_h v_h) \, dx - b((v_h,\psi_h),\lambda)$$

The term $b((v_h, \psi_h), \lambda)$ can be estimated as in Lemma 4.2. The stabilisation term is easily bounded using the Cauchy-Schwarz inequality

$$\int_{\Omega} (\phi - \Delta_h u) (\psi_h - \Delta_h v_h) \, dx \le \|\phi - \Delta_h u\|_{0,\Omega} \|\psi_h - \Delta_h v_h\|_{0,\Omega}$$

We further note that, for $u \in H^2(\Omega)$,

$$\int_{\Omega} \Delta_h u v_h \, dx = -\int_{\Omega} \nabla u \cdot \nabla v_h \, dx = \int_{\Omega} \Delta u v_h \, dx, \quad v_h \in S_{h,0}^{k+1},$$

and thus

$$\Delta_h u = \Pi_h^{k+1} \phi$$

where Π_h^{k+1} is the $L^2(\Omega)$ -orthogonal projection onto $S_{h,0}^{k+1}$. The proof follows by using the approxi-

mation property (3.1) on S_h^{k+1} . The following lemma estimates the best approximation error in the mesh-dependent norm. LEMMA 4.8. Let $(u, \phi) \in \mathcal{V}$ with $u \in W^{k+1,p}(\Omega)$ (for $p \geq 2$) and $\phi \in H^{k+1}(\Omega)$. Then, there exists an element $(w_h, \psi_h) \in \mathcal{V}_h^k$ such that

$$|(u - w_h, \phi - \psi_h)|_{k,h} \le C \left(h^k ||u||_{k+1,\Omega} + h^k ||\phi||_{k+1,\Omega} + h^{k-\frac{1}{2}-\frac{1}{p}} ||u||_{k+1,p,\Omega} \right).$$
(4.10) eqn(

Proof. We start with the definition of the mesh-dependent norm

$$|(u - w_h, \phi - \psi_h)|_{k,h}^2 = ||\nabla(\phi - \psi_h)||_{0,\Omega}^2 + ||\phi - \psi_h - \Delta_h (u - w_h)||_{0,\Omega}^2.$$

Let $R_h^{k+1}: H^1_0(\Omega) \to S_{h,0}^{k+1}$ be the Ritz-projector defined for $w \in H^1_0(\Omega)$

$$\int_{\Omega} \nabla (R_h^{k+1} w - w) \cdot \nabla v_h \, dx = 0, \ v_h \in S_{h,0}^{k+1}$$

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lemma5

lemma0

With $w_h = R_h^{k+1}u$, Property [P2] enables us to define $\psi_h \in S_{h,0}^k$ by

$$\int_{\Omega} \nabla w_h \cdot \nabla \mu_h + \psi_h \, \mu_h \, dx = 0, \ \mu_h \in (M_{bc})_h^k$$

Hence $(w_h, \psi_h) \in \mathcal{V}_h^k$ and, since $(u, \phi) \in \mathcal{V}$ and $(M_{bc})_h^k \subset M_{bc}$, we obtain

$$\int_{\Omega} \nabla(u - w_h) \cdot \nabla \mu_h + (\phi - \psi_h) \,\mu_h \, dx = 0, \ \mu_h \in (M_{bc})_h^k.$$

$$(4.11) \qquad \text{leq1}$$

We now use a triangle inequality to write

$$\begin{aligned} |(u - w_h, \phi - \psi_h)|_{k,h}^2 &= \|\phi - \psi_h - \Delta_h (u - w_h)\|_{0,\Omega}^2 + |\phi - \psi_h|_{1,\Omega}^2 \\ &\leq \|\phi - \psi_h\|_{1,\Omega}^2 + \|\Delta_h (u - w_h)\|_{0,\Omega}^2 \\ &= \|\Delta_h (u - w_h)\|_{0,\Omega}^2 + \|\phi - Q_h\phi\|_{1,\Omega}^2 + \|Q_h\phi - \psi_h\|_{1,\Omega}^2, \end{aligned}$$

where Q_h is a quasi-projection operator onto $S_{h,0}^k$ defined by

$$\int_{\Omega} Q_h \phi \, \mu_h \, dx = \int_{\Omega} \phi \, \mu_h \, dx, \ \mu_h \in (M_{bc})_h^k.$$

As above, Q_h is well-defined due to Assumption [P2]. First we estimate the term $\|\Delta_h(u-w_h)\|_{0,\Omega}$. By definition (3.5) of Δ_h and by choice $w_h = R_h^{k+1}u$,

$$\begin{split} \|\Delta_h(u-w_h)\|_{0,\Omega} &= \sup_{v_h \in S_{h,0}^{k+1}} \frac{\int_{\Omega} \Delta_h(u-w_h) v_h \, dx}{\|v_h\|_{0,\Omega}} \\ &= \sup_{v_h \in S_{h,0}^{k+1}} \frac{-\int_{\Omega} \nabla(u-w_h) \cdot \nabla v_h \, dx}{\|v_h\|_{0,\Omega}} = 0 \end{split}$$

We know that $Q_h \phi$ [20, 23] has the desired approximation property

$$|\phi - Q_h \phi|_{1,\Omega} \le Ch^k |\phi|_{k+1,\Omega}.$$

Hence we are left with the term $||Q_h\phi - \psi_h||_{1,\Omega}$. We start with an inverse estimate and use Assumption [P2] and (4.11) to get

$$\begin{aligned} \|\psi_h - Q_h\phi\|_{1,\Omega} &\leq \frac{C}{h} \|\psi_h - Q_h\phi\|_{0,\Omega} \leq \frac{C}{h} \sup_{\mu_h \in (M_{bc})_h^k} \frac{\int_{\Omega} (\psi_h - Q_h\phi)\,\mu_h\,dx}{\|\mu_h\|_{0,\Omega}} \\ &\leq \frac{C}{h} \sup_{\mu_h \in (M_{bc})_h^k} \frac{\int_{\Omega} (\psi_h - \phi)\,\mu_h\,dx}{\|\mu_h\|_{0,\Omega}} \\ &\leq \frac{C}{h} \sup_{\mu_h \in (M_{bc})_h^k} \frac{\int_{\Omega} \nabla(u - w_h) \cdot \nabla\mu_h\,dx}{\|\mu_h\|_{0,\Omega}}. \end{aligned}$$

Since w_h is the Ritz projection of u onto $S_{h,0}^{k+1}$, the final result follows by using Lemma 4.9. The following lemma is proved by using the ideas in [16, Lemma 3.2]. See also [29]. LEMMA 4.9. Let $k \in \mathbb{N}$ and $p \in \mathbb{R}$ such that $k \geq 1$ and $2 \leq p \leq \infty$. Let $R_h^{k+1} : H_0^1(\Omega) \to S_{h,0}^{k+1}$ be the Ritz projection as defined in Lemma 4.8. Then, there exists a constant C > 0 such that, for any $w \in W^{k+1,p}(\Omega) \cap H_0^1(\Omega)$,

$$\sup_{\substack{\mu_h \in (M_{bc})_h^k}} \frac{\int_{\Omega} \nabla(w - R_h^{k+1}w) \cdot \nabla\mu_h \, dx}{\|\mu_h\|_{0,\Omega}} \le Ch^{k + \frac{1}{2} - \frac{1}{p}} \|w\|_{k+1,p,\Omega}.$$
(4.12) [lest]

Proof. Let \mathcal{T}_h^1 be the set of elements in \mathcal{T}_h touching the boundary of Ω . Let $\mu_h \in (M_{bc})_h^k$ be arbitrary and $m_h \in S_{h,0}^k$ which coincides with μ_h at all interior finite element nodes. Since $S_{h,0}^k \subset S_{h,0}^{k+1}$, we have

$$\int_{\Omega} \nabla (w - R_h^{k+1} w) \cdot \nabla \psi_h \, dx = 0, \quad \psi_h \in S_{h,0}^k.$$

Thus we have

$$\int_{\Omega} \nabla(w - R_h^{k+1}w) \cdot \nabla\mu_h \, dx = \sum_{T \in \mathcal{T}_h^1} \int_T \nabla(w - R_h^{k+1}w) \cdot \nabla(\mu_h - m_h) \, dx.$$

The rest of the proof is exactly as in [16, Lemma 3.2]. \Box

5. Numerical Results. In this section, we show some numerical experiments for the sixth order elliptic equation using both types of boundary conditions. We compute the convergence rates in L^2 and semi H^1 -norms for u and ϕ , and the convergence rates in L^2 norms for our Lagrange multiplier. This computation will be done by using linear and quadratic finite element spaces.





(a) Initial mesh for simply supported boundary conditions



fig:init1

Fig. 5.1: Initial meshes

fig:init2

imesh

5.1. Simply supported boundary conditions.

Examples 1 and 2. We consider the exact solution

$$u = x^{5}(1-x)^{5}y^{5}(1-y)^{5} \text{ in } \Omega = (0,1)^{2}, \qquad (5.1) \quad \text{ex.sol.1}$$

for the first example and the exact solution

$$u = (e^y + e^x) x^5 (1-x)^5 y^5 (1-y)^5 \text{ in } \Omega = (0,1)^2, \qquad (5.2) \quad \text{ex.sol.2}$$

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lemma1

sec:numer

elem	$\frac{\ u - u_h\ _{0,\Omega}}{\ u\ _{0,\Omega}}$		$\frac{ u-u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\ \phi - \phi_h\ _{0,\Omega}}{\ \phi\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\ \lambda - \lambda_h\ _{0,\Omega}}{\ \lambda\ _{0,\Omega}}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
8	1.71e+02		1.44e + 02		$8.51e{+}01$		$4.33e{+}01$		1.88e+01	
32	5.17e + 01	1.72	$3.71e{+}01$	1.96	1.85e+01	2.20	8.14	2.41	3.15	2.58
128	1.74e + 01	1.57	1.18e+01	1.65	5.59	1.72	2.40	1.76	1.00	1.65
512	4.71	1.88	3.17	1.90	1.48	1.92	6.74 e- 01	1.83	2.86e-01	1.81
2048	1.20	1.97	8.11e-01	1.97	3.76e-01	1.98	2.05e-01	1.72	7.48e-02	1.94
8192	3.02e-01	1.99	2.08e-01	1.96	9.44e-02	1.99	7.65e-02	1.42	1.89e-02	1.98
32768	7.57e-02	2.00	5.59e-02	1.89	2.36e-02	2.00	3.42e-02	1.16	4.75e-03	2.00
131072	1.89e-02	2.00	1.74e-02	1.69	5.91e-03	2.00	1.65e-02	1.05	1.19e-03	2.00
524288	4.73e-03	2.00	6.75e-03	1.37	1.48e-03	2.00	8.20e-03	1.01	2.94e-04	2.00

Table 5.1: Discretisation errors for the simply supported boundary condition: linear case and exact solution (5.1)

ss1:linear

Table 5.2: Discretisation errors for the simply supported boundary condition: quadratic case and exact solution (5.1)

elem	$\frac{\ u - u_h\ _{0,\Omega}}{\ u\ _{0,\Omega}}$		$rac{ u-u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\left\ \phi-\phi_{h}\right\ _{0,\Omega}}{\left\ \phi\right\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\left\ \lambda - \lambda_h\right\ _{0,\Omega}}{\left\ \lambda\right\ _{0,\Omega}}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
8	1.01e-01		4.70e-01		2.13		1.00e+01		4.85e+01	
32	3.56e-04	8.15	3.32e-03	7.15	3.10e-02	6.10	4.27e-01	4.55	4.72	3.36
128	5.55e-05	2.68	6.31e-04	2.40	4.24e-03	2.87	9.65e-02	2.15	7.38e-01	2.68
512	4.15e-06	3.74	1.33e-04	2.24	3.63e-04	3.55	2.43e-02	1.99	8.15e-02	3.18
2048	2.89e-07	3.84	3.29e-05	2.02	3.13e-05	3.54	6.26e-03	1.96	9.36e-03	3.12
8192	2.24e-08	3.69	8.23e-06	2.00	3.22e-06	3.28	1.58e-03	1.99	1.14e-03	3.04
32768	2.15e-09	3.38	2.06e-06	2.00	3.76e-07	3.10	3.96e-04	2.00	1.41e-04	3.01

s1:quadratic

for the second example, where both functions satisfy simply supported boundary conditions $u = \Delta u = \Delta^2 u = 0$ on $\partial \Omega$. We start with the initial mesh as given in the left picture of Figure 5.1, and we compute the relative error in various norms associated with our variables at each step of refinement.

From Tables 5.1 and 5.3, we can see the quadratic convergence of errors in L^2 -norms of the linear finite element method for u, ϕ and λ , whereas the convergence of errors in the semi H^1 -norm for uis quadratic slightly better than linear but for ϕ it is linear. We note that convergence in the semi H^1 -norm for u is better in the earlier steps of refinement and as the refinement becomes finer and finer, the convergence rate becomes almost linear.

We have tabulated numerical results with the quadratic finite element method in Tables 5.2 and 5.4. Working with the quadratic finite element we see slightly better than $O(h^3)$ rate of convergence for the convergence of the errors in L^2 -norm for u, whereas the convergence is of $O(h^2)$ for the errors in the semi H^1 -norm. Similarly, the errors in the L^2 -norm for ϕ and λ converge with order $O(h^3)$, respectively, whereas the errors in the semi H^1 -norm of ϕ converge with $O(h^2)$. The numerical results follow the predicted theoretical rates also for both examples.

1	$\frac{\ u-u_h\ _{0,\Omega}}{\ u\ }$		$\frac{ u-u_h _{1,\Omega}}{ u }$		$\frac{\ \phi-\phi_h\ _{0,\Omega}}{\ \phi\ }$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\ \lambda - \lambda_h\ _{0,\Omega}}{\ \lambda\ }$	
elem	$\ u\ _{0,\Omega}$		$ u _{1,\Omega}$		$\ \varphi \ _{0,\Omega}$		$ \varphi _{1,\Omega}$		$\ \Lambda\ _{0,\Omega}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
8	1.77e + 02		1.49e + 02		8.84e + 01		4.50e+01		1.95e+01	
32	6.18e + 01	1.52	4.44e + 01	1.75	2.21e+01	2.00	9.75	2.21	3.76	2.38
128	1.98e+01	1.64	1.35e+01	1.72	6.40	1.79	2.74	1.83	1.13	1.74
512	5.28	1.91	3.55	1.93	1.66	1.94	7.46e-01	1.88	3.16e-01	1.83
2048	1.34	1.98	9.04e-01	1.98	4.20e-01	1.98	2.21e-01	1.76	8.21e-02	1.94
8192	3.36e-01	1.99	2.30e-01	1.97	1.05e-01	2.00	7.94e-02	1.47	2.07e-02	1.98
32768	8.42e-02	2.00	6.12e-02	1.91	2.64e-02	2.00	3.48e-02	1.19	5.20e-03	2.00
131072	2.10e-02	2.00	1.85e-02	1.73	6.59e-03	2.00	1.67 e-02	1.06	1.30e-03	2.00
524288	5.26e-03	2.00	6.94e-03	1.41	1.65e-03	2.00	8.27e-03	1.02	3.25e-04	2.00

Table 5.3: Discretisation errors for the simply supported boundary condition: linear case and exact solution (5.2)

ss2:linear

Table 5.4: Discretisation errors for the simply supported boundary condition: quadratic case and exact solution (5.2)

elem	$\frac{\ u - u_h\ _{0,\Omega}}{\ u\ _{0,\Omega}}$		$\frac{ u-u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\ \phi-\phi_h\ _{0,\Omega}}{\ \phi\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\ \lambda - \lambda_h\ _{0,\Omega}}{\ \lambda\ _{0,\Omega}}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
8	3.07e-01		1.43		6.47		3.06e+01		1.53e+02	
32	4.05e-03	6.24	2.10e-02	6.09	1.33e-01	5.60	1.51	4.34	1.71e+01	3.16
128	3.25e-04	3.64	2.44e-03	3.11	1.55e-02	3.11	3.27e-01	2.21	2.69	2.67
512	2.20e-05	3.89	4.51e-04	2.44	1.29e-03	3.59	8.19e-02	2.00	3.06e-01	3.13
2048	1.44e-06	3.93	1.10e-04	2.04	1.09e-04	3.56	2.11e-02	1.95	3.59e-02	3.09
8192	1.01e-07	3.84	2.75e-05	2.00	1.10e-05	3.30	5.34e-03	1.98	4.39e-03	3.03
32768	8.32e-09	3.59	6.87e-06	2.00	1.28e-06	3.10	1.34e-03	2.00	5.46e-04	3.01

s2:quadratic

Example 3. In the third example we consider the exact solution satisfying $u = 0, \Delta u = 0, \Delta^2 u = 0$ but $\nabla u \cdot \mathbf{n} \neq 0$ on the boundary:

$$u = \sin(\pi x) \sin(\pi y)$$
 in $\Omega = (0, 1)^2$. (5.3)

We note that the exact solutions chosen for Examples 1 and 2 satisfy $\nabla u \cdot \mathbf{n} = 0$ on the boundary of the domain Ω .

We start with the initial mesh as given in the left picture of Figure 5.1 and compute the relative error in various norms for all three variables at each step of refinement. The computed errors in different norms are tabulated in Tables 5.5 and 5.6. Interestingly, we still get the same rate of convergence for most of the norms with two exceptions: (i) in case of the linear finite element method, we do not observe a super-convergence rate in H^1 -norm of u, and (ii) in the quadratic finite element method, the rate of convergence in L^2 -norm of u is only $O(h^3)$.

5.2. Clamped boundary conditions.

Example 1. We choose the exact solution

$$u = 4096x^3(1-x)^3y^3(1-y)^3 \text{ in } \Omega = (0,1)^2, \tag{5.4}$$

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elem	$\frac{\ u-u_h\ _{0,\Omega}}{\ u\ _{0,\Omega}}$		$\frac{ u - u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\ \phi - \phi_h\ _{0,\Omega}}{\ \phi\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\ \lambda - \lambda_h\ }{\ \lambda\ _{0,\mathfrak{G}}}$	$\frac{0,\Omega}{\Omega}$
	error	rate	error	rate	error	rate	error	rate	error	rate
8	8.42e-01		8.78e-01		7.47e-01		1.23e+03		5.95e-01	
32	4.22e-01	1.00	4.91e-01	0.84	3.30e-01	1.18	6.65e + 02	0.89	2.27e-01	1.39
128	1.32e-01	1.68	2.18e-01	1.17	9.83e-02	1.75	3.13e+02	1.08	6.42e-02	1.83
512	3.50e-02	1.91	1.01e-01	1.11	2.57e-02	1.93	1.52e + 02	1.04	1.65e-02	1.95
2048	8.88e-03	1.98	4.95e-02	1.03	6.50e-03	1.98	$7.53e{+}01$	1.01	4.16e-03	1.99
8192	2.22e-03	1.99	2.46e-02	1.01	1.63e-03	2.00	$3.76e{+}01$	1.00	1.04e-03	2.00
32768	5.58e-04	2.00	1.23e-02	1.00	4.08e-04	2.00	1.87e + 01	1.00	2.61e-04	2.00

Table 5.5: Discretisation errors for the simply supported boundary condition: linear case and exact solution (5.3)

ss3:linear

Table 5.6: Discretisation errors for the simply supported boundary condition: quadratic case and exact solution (5.3)

elem	elem $\frac{\ u-u_h\ _{0,\Omega}}{\ u\ _{0,\Omega}}$		$rac{ u-u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\left\ \phi-\phi_{h}\right\ _{0,\Omega}}{\left\ \phi\right\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\ \lambda - \lambda_h\ _{0,\Omega}}{\ \lambda\ _{0,\Omega}}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
8	2.12e-01		2.29e-01		1.63e-01		2.90e+02		1.14e-01	
32	2.12e-02	3.33	4.22e-02	2.44	1.70e-02	3.26	6.20e+01	2.22	1.36e-02	3.08
128	1.98e-03	3.42	9.99e-03	2.08	1.78e-03	3.26	1.52e+01	2.03	1.63e-03	3.05
512	2.14e-04	3.21	2.49e-03	2.01	2.07e-04	3.11	3.80	2.00	2.01e-04	3.02
2048	2.54e-05	3.07	6.21e-04	2.00	2.52e-05	3.03	9.51e-01	2.00	2.51e-05	3.00
8192	3.14e-06	3.02	1.56e-04	2.00	3.14e-06	3.01	2.38e-01	2.00	3.14e-06	3.00

s3:quadratic

so that the exact solution satisfies the clamped boundary condition

$$u = \Delta u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial \Omega$$

For our clamped boundary condition we start with the initial mesh as given in the right picture of Figure 5.1. In the following ϕ and λ are discretised using the linear finite element space, whereas u is discretised using the quadratic finite element space. That means we use the finite element spaces with k = 1. The numerical results are tabulated in Table 5.7. In this example, we get higher convergence rates than predicted by the theory for all errors. These results seem to indicate a higher order of convergence for u in the semi H^1 -norm than in the L^2 -norm. This can either be due to the asymptotic rates not being achieved at the grid levels considered, or to some genuine super-convergence result. Understanding this phenomenon in more depth is the purpose of future work. As in the case of Examples 1 and 2 of the simply supported boundary condition we see the better convergence rates for the semi H^1 -norm in earlier steps of refinement. However, when we refine further the convergence rates achieved at the earlier steps of refinement.

elem	$\frac{\left\ u - u_h\right\ _{0,\Omega}}{\left\ u\right\ _{0,\Omega}}$		$rac{ u-u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\left\ \phi-\phi_{h}\right\ _{0,\Omega}}{\left\ \phi\right\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\left\ \lambda - \lambda_h\right\ _{0,\Omega}}{\left\ \lambda\right\ _{0,\Omega}}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
32	4.34		8.51		7.47e-01		9.69e-01		9.20e-01	
128	1.09	2.00	3.46	1.30	3.06e-01	1.29	5.38e-01	0.85	3.88e-01	1.25
512	1.85e-01	2.55	6.43e-01	2.43	1.26e-01	1.29	2.86e-01	0.91	2.13e-01	0.86
2048	2.43e-02	2.93	7.73e-02	3.06	2.38e-02	2.40	1.30e-01	1.14	9.34e-02	1.19
8192	4.76e-03	2.35	9.94e-03	2.96	4.49e-03	2.40	6.35e-02	1.03	2.40e-02	1.96
32768	1.11e-03	2.11	1.39e-03	2.84	1.02e-03	2.13	3.17e-02	1.00	4.67 e-03	2.36
131072	2.73e-04	2.02	2.74e-04	2.34	2.53e-04	2.02	1.58e-02	1.00	8.24e-04	2.50

Table 5.7: Discretisation errors for the clamped boundary condition: exact solution (5.4)

Example 2. For our last example with clamped boundary condition the exact solution is chosen

$$u = 4096x^{3}(1-x)^{3}y^{3}(1-y)^{3}\left(\frac{2}{5}e^{x} + \cos(y)\right).$$
(5.5) [ex.sol.4]

As in the previous example, this solution also satisfies the clamped boundary condition. We have tabulated the relative error in various norms in Table 5.8. The results are very similar to the ones as in the first example. However, the relative error in the case of clamped boundary conditions are higher than in the case of simply supported boundary conditions. We can also see that the asymptotic rates of error reduction start later in this case due to the extrapolation on the boundary patch of the domain.

elem	$\frac{\ u-u_h\ _{0,\Omega}}{\ u\ _{0,\Omega}}$		$rac{ u-u_h _{1,\Omega}}{ u _{1,\Omega}}$		$\frac{\ \phi-\phi_h\ _{0,\Omega}}{\ \phi\ _{0,\Omega}}$		$\frac{ \phi - \phi_h _{1,\Omega}}{ \phi _{1,\Omega}}$		$\frac{\ \lambda - \lambda_h\ _{0,\Omega}}{\ \lambda\ _{0,\Omega}}$	
	error	rate	error	rate	error	rate	error	rate	error	rate
32	8.38		1.33e+01		7.77e-01		1.00		1.19	
128	1.36	2.63	4.01	1.73	4.45e-01	0.80	6.54e-01	0.62	6.40e-01	0.90
512	2.05e-01	2.72	7.61e-01	2.40	1.30e-01	1.78	2.89e-01	1.18	2.20e-01	1.54
2048	2.46e-02	3.06	8.75e-02	3.12	2.40e-02	2.44	1.30e-01	1.15	9.39e-02	1.23
8192	4.76e-03	2.37	1.07e-02	3.03	4.53e-03	2.40	6.36e-02	1.03	2.41e-02	1.96
32768	1.10e-03	2.11	1.45e-03	2.89	1.03e-03	2.13	3.17e-02	1.00	4.71e-03	2.36
131072	2.72e-04	2.02	2.78e-04	2.38	2.55e-04	2.02	1.59e-02	1.00	8.35e-04	2.50

Table 5.8: Discretisation errors for the clamped boundary condition: exact solution (5.5)

clbc2

clbc1

 \mathbf{as}

REMARK 5.1. We have proved the error estimate in the mesh-dependent norm $|\cdot|_{k,h}$ for the clamped boundary condition case. This norm can be estimated by the standard L^2 -norm and H^1 -norm as follows. By using the triangle inequality we have

$$\begin{aligned} |(u - u_h, \phi - \phi_h)|_{k,h}^2 &= \|\phi - \phi_h - \Delta_h u + \Delta_h u_h\|_{0,\Omega}^2 + \|\nabla(\phi - \phi_h)\|_{0,\Omega}^2 \\ &\leq C(\|\phi - \phi_h\|_{0,\Omega}^2 + \|\Delta_h u - \Delta_h u_h\|_{0,\Omega}^2 + \|\nabla(\phi - \phi_h)\|_{0,\Omega}^2) \end{aligned}$$

We now only consider the middle term of the last line of the last inequality. Using the definition of Δ_h , the L^2 -norm and the standard inverse estimate we have a constant C independent of the mesh-size

h such that

$$\begin{aligned} \|\Delta_h u - \Delta_h u_h\|_{0,\Omega} &\leq \sup_{\phi_h \in S_{h,0}^{k+1}} \frac{\int_{\Omega} (\nabla u - \nabla u_h) \cdot \nabla \phi_h \, dx}{\|\phi_h\|_{0,\Omega}} \\ &\leq \frac{C}{h} \|\nabla u - \nabla u_h\|_{0,\Omega}. \end{aligned}$$

Since the computed errors behave like $\|\phi - \phi_h\|_{0,\Omega} = O(h^2)$, $\|\nabla u - \nabla u_h\|_{0,\Omega} = O(h^2)$ and $\|\nabla \phi - \nabla \phi_h\|_{0,\Omega} = O(h)$, the errors for u and ϕ in the mesh-dependent norm $|\cdot|_{k,h}$ behave as $|(u - u_h, \phi - \phi_h)|_{k,h} = O(h)$.

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Appendix A. Proof of Theorem 2.1.

The existence and uniqueness of a solution to (2.6) follows from the Ladyzenskaia–Babushka– Brezzi theory, provided that we establish the following properties.

- 1. The bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and the linear form $\ell(\cdot)$ are continuous on $V \times V$, $V \times M_{bc}$ and V, respectively.
- 2. The bilinear form $a(\cdot, \cdot)$ is coercive on the kernel space

$$\mathcal{V} = \{ (v, \psi) \in V : b((v, \psi), \mu) = 0, \ \mu \in M_{bc} \}.$$

3. The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition, for some $\beta > 0$:

$$\inf_{\mu \in M_{bc}} \sup_{(v,\psi) \in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V \|\mu\|_{M_{bc}}} \ge \beta.$$

The Cauchy-Schwarz inequality implies that the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and the linear form $\ell(\cdot)$ are continuous on $V \times V$, $V \times M_{bc}$ and V, respectively. We now turn our attention to the second condition. In fact, for $(u, \phi) \in V$ satisfying $b((u, \phi), \mu) = 0$ for all $\mu \in M_{bc} \supset H_0^1(\Omega)$ we have with $\mu = u$

$$\int_{\Omega} \nabla u \cdot \nabla u \, dx = -\int_{\Omega} \phi u \, dx.$$

Hence using Cauchy-Schwarz and the Poincaré inequality we find

$$\|\nabla u\|_{0,\Omega}^2 \le C \|\phi\|_{0,\Omega} \|\nabla u\|_{0,\Omega}.$$

Thus we have

$$\|\nabla u\|_{0,\Omega} \le C \|\phi\|_{0,\Omega}.$$

From this inequality we infer

$$\|\nabla u\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2 \le C \|\phi\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2.$$

We use the Poincaré inequality again to obtain the coercivity

$$\|\nabla u\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2 \le C \|\phi\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2 \le Ca((u,\phi),(u,\phi)), \quad (u,\phi) \in \mathcal{V}.$$

Let us now consider the inf-sup condition in the case of simply supported BCs, that is M_{bc} = $H_0^1(\Omega)$ with natural norm. For all $\mu \in H_0^1(\Omega)$,

$$b((\mu, 0), \mu) = -\langle \mu, \Delta \mu \rangle = \int_{\Omega} |\nabla \mu|^2 = \|\mu\|_{H^1_0(\Omega)}^2$$

and thus

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} \geq \frac{b((\mu,0),\mu)}{\|\mu\|_{H^1_0(\Omega)}} \geq \|\mu\|_{H^1_0(\Omega)}.$$

We finally consider the inf-sup condition in the case of clamped boundary conditions, that is $M_{bc} = \{\mu \in H^{-1}(\Omega) : \Delta \mu \in H^{-1}(\Omega)\}$ with corresponding graph norm. We have

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} = \sup_{(v,\psi)\in V} \frac{\langle\psi,\mu\rangle - \langle v,\Delta\mu\rangle}{\|(v,\psi)\|_V}$$

Now setting $\psi = 0$ we obtain

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} \ge \sup_{v\in H_0^1(\Omega)} \frac{\langle v,\Delta\mu\rangle}{\|\nabla v\|_{0,\Omega}} \ge c_1 \|\Delta\mu\|_{-1,\Omega},$$

where we have used Poincaré inequality in the last step. Similarly, using v = 0 we get

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} \ge \sup_{\psi\in H^1_0(\Omega)} \frac{\langle\psi,\mu\rangle}{\|\nabla\psi\|_{0,\Omega}} \ge c_2 \|\mu\|_{-1,\Omega},$$

and hence there exists a constant $\beta>0$ such that

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} = \sup_{(v,\psi)\in V} \frac{\langle\psi\,\mu\rangle - \langle v,\Delta\mu\rangle}{\|(v,\psi)\|_V} \ge \beta \|\mu\|_{M_{bc}}.$$

Hence (2.6) has a unique solution.

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